Of Physics and Factory Physics

Unabridged Version 1.02

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Abstract

The “Principle of Least Action” is the foundational principle of fundamental physics. Application of this principle to the supply chain naturally results in an “uncertainty principle” linking variation in production to variation in net-inventory. The model also provides an easy means of determining the stationary distribution of net-inventory for a variety of control strategies. The formalism results in a control strategy that outperforms commonly used control methods.
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1 Introduction

Factory Physics is a text book by Hopp and Spearman [10] that first appeared in 1996 and sought to provide a systematic framework to describe the behavior of manufacturing systems. Like “real” physics, factory physics is stated as a set of “laws” about the way production-inventory systems behave. Unlike “real” physics, the laws of factory physics do not follow from a single unifying framework.

Lord Rutherford, discoverer of the nucleus in 1911, once stated that, “All science is either physics or stamp collecting” implying that the only true science was physics and all the others were simply cataloging various observations.[1] Alas, it appears that factory physics has progressed little beyond the stamp collecting stage.

The purpose of this paper is to move toward a more general framework that will describe the behavior of production-inventory systems. By doing this, we will obtain further insights into how manufacturing systems behave and even suggest better ways to design and control them.

[1] This was probably not true 100 years ago and is certainly untrue today. Ironically, Rutherford received his Nobel Prize in chemistry, not physics.

One particular factory physics law deals with variability—the greater the variability the poorer the performance. The basic model presented there postulates that the only way that production and demand can be synchronized is to employ some sort of "clutch" or "buffer" between the two and that there are only three possibilities: inventory, time, and/or capacity. The inventory buffer represents any inventory between production and demand. Likewise, the time buffer represents any time between the arrival of a customer and when the demand is satisfied. The capacity buffer is defined to be the difference between available capacity and the average demand.

A vertically integrated "configure-to-order" (CTO) operation provides a good example of how these three buffers interact. Such systems usually wait until a customer gives a, possibly, unique specification and then configures (or assembles) the final product to that specification. Consequently, the system employs both a time buffer between the CTO operation and the customer and an inventory buffer between the final assembly process and fabrication (or a supplier). In the later case, the assembly process represents "demand" and the fabrication (supplier) provides the "production".

However, if such a system were to use only an inventory buffer it would require stock for each and every possible configuration of which there can be an enormous number. If it were to employ only a time buffer, the customer would have to wait not only the time for the desired item to be configured but also the time it would take to acquire the the components. By using a combination, inventory of components, and time to configure, the system has neither excessive inventory nor lengthly wait times.

The capacity buffer affects both the inventory and time buffers. With a significant amount of capacity above the average demand, the fabrication time can be kept low. However, if there is not much extra capacity, this queue can become very long requiring a longer wait and,typically, more inventory.

So it is clear that these three buffers are interrelated. The question is how are they related and how can that relation be generalized.

1.1 This Paper

This paper is addressed to the operations researcher interested in supply chain applications. Because it will consider some fundamental models from physics, the paper will necessarily involve some tutorial. This will assume the reader is mathematically sophisticated but has little or no knowledge of fundamental physics. We begin with a "basic" model of a production-inventory system that models a base stock system in the presence of demand variation. Using Itô calculus we derive a relation between the variability in the net-inventory and production. The form of this result is similar to a fundamental relation found in quantum physics which motivates the remainder of the paper.

We then consider the fundamental model that underlies almost all branches of physics, from classical mechanics to quantum field theory—The Principle of Least Action, also known as Hamilton's Principle [25]. Adding randomness and then applying a "potential" to control the "motion" of an inventory "particle" we arrive at the following conclusions:

1. The "buffers" mentioned above are linked in a "canonical" way using the Hamiltonian Least Action formalism.

2. There is an "uncertainty principle" between production and net inventory (i.e., as the variance of one decreases, the other must increase).

3. We can use a "potential" to set the control of a production-inventory system.

4. We can easily write down the final (steady state) joint distribution of net-flow and net-inventory in terms of the system "Hamiltonian."

5. The base stock model is a *special case* of the Least Action model. Consequently, the Least Action model cannot perform worse than the traditional model and will, in many cases, perform better.

2 The Basic Production-Inventory Model

We start with a basic production-inventory model that can be written as

$$Q(t + \Delta t) = Q(t) + X(t) - D(t)$$

where Q represents net-inventory, X represents production during period t, a decision variable, and D represents demand, usually random, with a mean of $\lambda \Delta t$ during the same period. In general, we can define

$$D(t) = \lambda \Delta t + W(\Delta t)$$

where λ is the mean of the demand and W is a random variable with zero mean and a variance that depends on Δt. We also define,

$$\begin{aligned} X(t) &= (\lambda + g(Q(t)))\Delta t & (1) \\ &= R(t)\Delta t & (2) \end{aligned}$$

where g is some real function of the net-inventory. Combining these yields,

$$\Delta Q(t) = g(Q(t))\ \Delta t + W(\Delta t) \qquad (3)$$

This basic model exhibits the behavior of many production systems including a base stock system, a reorder point/reorder quantity system, and even a time-phased reorder point system that seeks to keep inventory at a given level (safety stock).

A common version of this model is the base stock model,

$$\Delta Q(t) = -g(Q(t) - Q_0)\Delta t + W(\Delta t) \qquad (4)$$

where Q_0 is a target inventory level (here set equal to zero without loss of generality) and g is a constant and is usually set equal to one. This model also represents a time-phased reorder point (i.e., material requirements planning) model with a lot-for-lot order sizing rule.

We can model the transient and steady state behavior of this system using a stochastic differential equation.

2.1 Formulation Using Stochastic Differential Equations

Before modeling the system in (4), let us consider an equation with demand only,

$$\begin{aligned} Q(t+\Delta t) &= Q(t) - D(t) \\ &= Q(t) - \lambda \Delta t + W(\Delta t) \end{aligned}$$

Under certain conditions (that will be specified below) we can model the dynamics of such as system using a stochastic differential equation (see, e.g., [19]) as,

$$dQ = -\lambda dt + \sigma dW$$

and let $Q(0) = 0$. The solution is

$$Q(t) = \lambda t + \sigma \int_0^t dW$$

where the integral is a random variable known as an *Itô integral*. The expectation will be simply λt and the variance can be computed using results from Itô calculus (see § 10.1),

$$\begin{aligned} \mathrm{var}(Q(t)) &= \left\langle \left(\sigma \int_0^t dW \right)^2 \right\rangle \\ &= \sigma^2 \int_0^t 1^2 ds \\ &= \sigma^2 t \end{aligned}$$

where $\langle X \rangle$ represents the expected value of X. In many real world cases we observe what is known as "quadratic variation" that implies that the variance of demand will increase linearly in time,

$$\lim_{t \to \infty} \frac{\mathrm{var}(Q(t))}{t} = \psi^2 \lambda$$

where ψ^2 is the variance to mean ratio of demand. Then,

$$\sigma^2 = \psi^2 \lambda$$

Then the SDE of the base stock model will be,

$$\begin{aligned} dQ &= -gQdt + \psi\sqrt{\lambda}\, dW \\ Q(0) &= Q_0 \end{aligned} \tag{5}$$

and is an example of the Ornstein-Uhlenbeck process whose solution is,

$$Q(t) = Q_0 e^{-gt} + \psi\sqrt{\lambda} \int_0^t e^{-g(t-s)} dW$$

The steady state mean and variance are obtained using relations in § 10.1 and allowing $t \to \infty$.

$$\begin{aligned} \langle Q \rangle &= 0 \\ \mathrm{var}(Q) &= \frac{\psi^2 \lambda}{2g} \end{aligned}$$

The variance of production required to keep the variance of Q at the level given above will be,

$$\begin{aligned} \mathrm{var}(R) &= g^2 \mathrm{var}(Q) \\ &= g\frac{\psi^2 \lambda}{2} \end{aligned} \tag{6}$$

If there is inherent variation in production then (6) will be a lower bound so that

$$\sigma_R \sigma_Q \geq \frac{\psi^2 \lambda}{2} \tag{7}$$

The implication is that if we want tighter control on inventory, we must allow production to vary more. This accommodation requires extra capacity above and beyond the average demand to be available at all times so it will be available when it is needed to bring up the inventory to the desired target. This translates to a larger capacity buffer. On the other hand, if we allow more variation in net-inventory, we can live with a smaller capacity buffer. Of course, more variation in inventory, implies more inventory and/or backorders, on average. These interactions are examples of the basic trade-off between the buffers introduced in § 1.

3 Relationship to Physics

The inequality (7) is reminiscent of a fundamental relation in quantum mechanics known as the Heisenberg Uncertainty Principle,

$$\sigma_p \sigma_q \geq \hbar/2$$

where p and q are the momentum and position, respectively. Such relations occur in physics only between special pairs of variables known as *canonical conjugates.*

In addition to the uncertainty relations of quantum mechanics, canonical conjugates have other, more fundamental, implications. The most important of these is Noether's theorem that relates the presence of a symmetry to a corresponding conservation law [18]. For instance, the fact that physics is the same everywhere predicts the conservation of momentum. That physics is the same today, yesterday, and tomorrow implies conservation of energy. It also works the other way. Physicists had long known that charge was conserved. Noether's result meant that there must be a corresponding symmetry that had been unobserved beforehand. This symmetry, "gauge invariance," was then identified and became important in the development of the field of quantum electrodynamics. It would be interesting to know if such symmetry-conservation relations exist in supply chain dynamics.

This symmetry typically appears in the *Lagrangian*[2] of a dynamical system. Since supply chains are dynamic in nature and given the relation in (7), it seems reasonable to write the Lagrangian of a supply chain and see if we can model a real supply chain with it.

[2]The Lagrangian of operations research apparently came from the application of constraints of motion (e.g., a hoop rolling without slipping) in the mechanical Lagrangian formulation that result in "Lagrange multipliers" that are essentially the same as those used on operations research.

3.1 A Supply Chain Lagrangian

We will use the same notation as before but with lower case variables.[3] We then define q to be a generalized coordinate denoting the net-inventory and its time derivative will be the net-flow, viz.

$$\dot{q} = x - d$$

In mechanics, the motion of the particle determines its kinetic energy while a *potential* determines how the particle will move. An example is the motion of the Earth within the gravitational potential of the Sun. If there were no potential, the Earth would move at a constant velocity in a straight line. But in the presence of a gravitational potential, the Earth orbits the Sun in a motion that forms an ellipse with the Sun at one of the foci.

Thus, control of net-inventory "particle" will be accomplished by a "potential" that generates a "force" on the particle to constrain its motion. In addition to forces that control the particle, the potential also contains elements that generate the force resulting in the demand of the system. Fortunately, the potential is a scaler so it is simple to add these components. Also, as in physics, it makes things more convenient if the potential is a function of only the coordinate and not time or the net-flow, $V(q)$.

The Lagrangian is defined to be the difference between the kinetic energy and the potential energy. Typically, the kinetic energy involves a mass term that will not be included here. Mass in physics is related to inertia and to gravity. While inventory particles are not affected by gravity they will have inertia in many situations such as when production rates are not continuously varied but set periodically. As we will see, our formulation matches these situations exactly—without a mass term. Consequently, to avoid unnecessary notation we will set all the masses to one and drop them from the subsequent development. While massless formulations are not common in physics, they do occur (e.g., a Lagrangian in quantum field theory can be written such that there is no explicit mass term). However, this will mean that the formulation developed here will *not* be appropriate for situations in which the control response *can* be varied instantaneously and continuously.

Thus, the Lagrangian involves the kinetic energy, $\dot{q}/2$, and the potential energy, V(q), and is written as

$$L(q, \dot{q}, t) = \frac{1}{2}\dot{q}^2 - V(q)$$

Then, for *any* generalized coordinate, a corresponding "momentum" is defined as

$$p = \frac{\partial L(q, \dot{q})}{\partial \dot{q}}$$

Defined in this manner, the momentum becomes the *canonical conjugate* of the generalized position, q, and in this case is equal to the **net flow**,

$$p = \dot{q} = x - d \tag{8}$$

[3]To simplify the notation we will use the same notation as that employed in control theory (and first developed by Newton) indicating a generalized coordinate with q, its time derivative as $\dot{q}$, and its second derivative as $\ddot{q}$. Moreover, we will not explicitly indicate the time dependence of q and its derivatives as it would make the notation clumsy.

3.2 The Hamiltonian and the Principle of Least Action

The overarching principle that governs the dynamics of the system is governed by the Principle of Least Action. Not only does this principle apply to mechanics, it also applies to electricity and magnetism, optics, thermodynamics, and even quantum physics. In addition to physics the Least Action principle has been applied to economics [22], control theory [21], medicine [4], biology [13, 23], and even robotics [20]. The Least Action formalism is the underlying framework of the Standard Model of particle physics. As such, it has been spectacularly successful in not only explaining observed phenomena but in also predicting new phenomena and new particles. In the summer of 2012, we saw the first observations of the Higgs boson [1] that was predicted the the Standard Model almost 50 years earlier. For this prediction, Englert and Higgs won the Nobel Prize in physics in October of 2013.

Mathematically, Hamilton's Principle is written as,

$$\delta S = \delta \int_{t_i}^{t_f} L(q, \dot{q}, t) dt = 0$$

where the δ indicates an infinitesimal variation in the path. This implies that the integral in any natural process will result in a *stationary* path, either a minimum or a maximum. Physical processes result in a minimum value.

Using the calculus of variations to find the Lagrangian function that minimizes the integral results in a necessary condition known as the *Euler-Lagrange Equation* (see [25] for a derivation).

$$\frac{d}{dt}\left(\frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}}\right) - \frac{\partial L(q, \dot{q}, t)}{\partial q} = 0 \tag{9}$$

In classical dynamics, force is defined as the time derivative of momentum, $\dot{p}$, we see from equation (9),

$$\dot{p} = \frac{\partial L(q, \dot{q})}{\partial q}$$

Thus, $\dot{p}$ (or the "force") will be the change in the net-flow per unit time. If the mean demand remains constant, this change will occur in the production rate. Thus, the potential provides a means to describe the control used in the production-inventory system.

Upon reflection, the principle of least action is astounding. If a ball is to roll down a hill, it will choose a path that minimizes the above integral. The problem here is not to simply find a point that minimizes a function but to find a *function* that minimizes an integral. Given the difficulty of the former, how does a simple physical system accomplish the latter? Nonetheless, it does and the principle of least action leads directly to many laws of physics including Newton's laws of motion, Schrödinger's equation, relativistic electrodynamics, and quantum field theory.

3.3 The Hamiltonian

Although the Lagrangian is written in terms of q and $\dot{q}$, we would like to be able to describe the system in terms of net-inventory, q, and net-flow, p. This can be done by applying a *Legendre transformation* and defining the *Hamiltonian* as

$$\mathcal{H}(q, p, t) = \dot{q}p - L(q, \dot{q}, t) \tag{10}$$

Or, rewritten in terms of q and p,

$$\mathcal{H}(q,p,t) = \frac{p^2(t)}{2} + V(q) \tag{11}$$

Taking the total differential of (10),

$$d\mathcal{H} = \dot{q}dp + pd\dot{q} - \frac{\partial L}{\partial \dot{q}}d\dot{q} - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial t}dt$$

Because $p = \partial L/\partial \dot{q}$ the second and third terms cancel leaving

$$d\mathcal{H} = \dot{q}dp - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial t}dt$$

But $\dot{p} = \partial L/\partial q$, so

$$d\mathcal{H} = \dot{q}dp - \dot{p}dq - \frac{\partial L}{\partial t}dt \tag{12}$$

Moreover, in general, $d\mathcal{H}$ can be written as

$$d\mathcal{H} = \frac{\partial \mathcal{H}}{\partial p}dp + \frac{\partial \mathcal{H}}{\partial q}dq + \frac{\partial \mathcal{H}}{\partial t}dt$$

Thus, for (12) to hold in general requires,

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial q} \tag{13}$$

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p} \tag{14}$$

$$\frac{\partial L}{\partial t} = -\frac{\partial \mathcal{H}}{\partial t} \tag{15}$$

of which (13) and (14) are known as the *canonical equations of Hamilton.* Typically, these equations do not offer an easier way to solve problems in classical dynamics. Instead Hamilton's equations are elegant in their simplicity and in their (slightly) broken symmetry (one can exchange q and p with only a sign change). They also offer deep insight into both classical and quantum mechanics. More familiarly, the first equation gives us the classic $F = ma$ (the potential is defined so that $F = -\partial V/\partial q$) while the second indicates the relation between the generalized velocity and the momentum.

A variant of the Least Action principle known as the "Maximum Principle" has been used in control theory for many years (see, e.g., [2]). For an excellent and more recent exposition, see Chapter 2 in Sethi and Thompson [21]. Instead of being minimized, an integral similar to the action integral is maximized using dynamic programming methods. Maximization is appropriate as it represents the profit of the control policy and is in keeping with the action principle which seeks an *extremum* be it a minimum or a maximum. They then derive a variation of the Hamiltonian that is the solution to the Hamilton-Jacobi equation and then use the construct to maximize the profit of a control trajectory. The "adjoint" variables used to denote marginal returns operate in the same way as canonical conjugates. However, the goal of this method is to determine optimal policies and not to model the particular dynamics of a given system.

3.4 Poisson Brackets

Hamilton's equations can be further generalized through the use of Poisson brackets defined as,

$$\{f, g\} = \sum_{i=1}^{N} \left\{ \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right\}$$

where N is the number of dimensions in the generalized coordinate system and f and g are functions of q_i, p_i, and t. Some useful properties of the Poisson brackets are

$$\begin{aligned} \{f, g\} &= -\{g, f\} \\ \{f, f\} &= 0 \\ \{q_i, q_j\} &= 0 \\ \{p_i, p_j\} &= 0 \\ \{q_i, p_j\} &= \delta_{ij} \end{aligned}$$

where δ_{ij} is the Kronecker delta. The last three properties are characteristic of all conjugate variables. From (8) we obtain,

$$\{q_i, x_j\} = \delta_{ij} \tag{16}$$

$$\{d_i, q_j\} = \delta_{ij} \tag{17}$$

so that x and d are also canonical conjugates with q.

The Poisson brackets provide a compact expression of Hamilton's equations,

$$\begin{aligned} \dot{q} &= \{q, \mathcal{H}\} \\ \dot{p} &= \{p, \mathcal{H}\} \end{aligned}$$

Moreover, the equations of motion can be succinctly written in terms of the Poisson bracket. If we replace g with $\mathcal{H}$ we obtain

$$\{\mathcal{H}, g\} = \sum_{i=1}^{N} \left\{ \frac{\partial \mathcal{H}}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial \mathcal{H}}{\partial p_i} \frac{\partial g}{\partial q_i} \right\}$$

But from Hamilton's equations, this is simply

$$\{\mathcal{H}, g\} = -\sum_{i=1}^{N} \left\{ \frac{\partial g}{\partial q_i} \dot{q} + \frac{\partial g}{\partial p_i} \dot{p} \right\}$$

So that,

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \frac{\partial g}{\partial t} - \{\mathcal{H}, g\}$$

For a more complete discussion, see, e.g., Goldstein, [8], p. 221, in particular equations (7.19) and (8.58). Thus, a quantity g is conserved if it is not explicitly time-dependent and if the Poisson bracket of the Hamiltonian is zero. The fact that the Hamiltonian "commutes" with itself leads to

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t} \tag{18}$$

So unless $\mathcal{H}$ is an *explicit* function of time, it represents a *constant of motion.* In Cartesian coordinates we see that the kinetic energy, $T = m\dot{q}^2/2$, thus

$$\begin{aligned}\mathcal{H} &= \dot{q}p - L \\ &= 2T - (T - V) \\ &= T + V\end{aligned}$$

so that $\mathcal{H}$ represents the total energy of the system.

3.5 Link between Classical and Quantum Mechanics

This formalism provides a direct link between classical dynamics and quantum mechanics. Under fairly general conditions, one can write the "commutator" of quantum mechanics in terms of the Poisson bracket of classical dynamics. The commutator is given by

$$[A, B] = AB - BA$$

where A and B are "operators" (e.g., Hermitian matrices for "observables"). The classical/quantum relation is then

$$[A, B] = i\hbar\{A, B\}$$

In quantum physics, canonical conjugates are related by the "Heisenberg Uncertainty Principle" that can be written in terms of the commutators,

$$\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}\langle[A, B]\rangle\right)^2$$

where $\langle A\rangle$ is the expected value of A. In the case of position and momentum, $[q, p] = i\hbar$ so that we obtain the more familiar "uncertainty principle" mentioned above.

$$\sigma_q\sigma_p \geq \frac{\hbar}{2}$$

The real reason for using generalized coordinates is that they need not represent physical quantities like position or momentum and that the Hamiltonian need not represent the total energy nor even anything with a physical analog. This makes it useful to model a supply chain.

4 Randomness and the Hamiltonian

While classical mechanics can conceivably model any system with an arbitrary number of particles, it can practically only be used to analyze relatively small systems. The study of thermodynamics allows us to study systems with a huge number of particles (on the order of 10^{23}). We will find a branch of thermodynamics called *statistical* physics most useful in that it provides us with a means to study random systems although it is not really random at all.

It was not until the quantum formulation of physics that randomness became an integral part of the description of Nature. Indeed, "quantum mechanics" simply *does not work* without a pair of probability density functions relating momentum and position. Nonetheless, the formulation that best fits the randomness found in production-inventory systems is closer to that found in statistical thermodynamics than in quantum mechanics.

We will follow the development of statistical physics and define a *phase space* with N ordered pairs, (q_i, p_i) for each particle i, where N is the number of particles. Each point in the phase space is known as a *microstate* with an associated probability and energy. Collections of microstates that have the same energy are called *macrostates* and we expect to find many microstates within a given macrostate. For example, the position and momentum of every molecule of the air in the room you are reading this paper represents one microstate. The temperature and pressure of the room define a macrostate. Clearly, there are *many* microstates in this macrostate!

The central assumption of statistical physics is that all states of the system with the same energy have the same probability of occurring. This is a direct consequence of the "Second Law of Thermodynamics" which states that entropy (disorder) will increase until it reaches a maximum which represents *equilibrium.* This maximum occurs when all accessible states have the same probability.

4.1 Liouville's Theorem

We define the density function of the phase space as $\rho(q, p; t)$ such that the expected value of a real function, $\langle g(t)\rangle$, is given by

$$\langle g(t)\rangle = \frac{\int_\Omega g(q,p)\rho(q,p;t)d\Omega}{\int_\Omega \rho(q,p;t)d\Omega}$$

where Ω is the set of all states. One of the fundamental results of statistical physics is,

$$\frac{d\rho}{dt} = 0$$

which implies that a given volume in phase space may change shape during its evolution but the volume will remain constant (see [12] for a proof). Using the Poisson bracket notation, we can write

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \{\rho, \mathcal{H}\} \tag{19}$$

which is known as "Liouville's Theorem" [16]. If the system is in equilibrium then $\frac{\partial\rho}{\partial t} = 0$ which implies $\{\rho, \mathcal{H}\} = 0$. One way to achieve this is to assume that ρ is independent of q and p. A more general way that incorporates Liouville's Theorem results in

$$\rho(q,p) = C\exp\left(-\frac{\mathcal{H}(q,p)}{kT}\right) \tag{20}$$

which can easily be verified using (19) assuming $\frac{\partial\rho}{\partial t} = 0$. Note that kT is the energy of the system having "temperature" T so that k, Boltzmann's constant, provides the proportionality constant between energy and temperature.

Thus, the steady state distribution of the net inventory and the net flow in a production-inventory system described by Hamiltonian $\mathcal{H}$ can be written down by simply knowing the Hamiltonian and the final *temperature* of the system. So the question remains as to what is the "temperature" of a supply chain?

4.2 The Equipartition Theorem

Another important result from statistical physics is the idea that the energy of a system of particles is divided equally among its various modes of energy. For example, a system of particles not under the influence of a potential (i.e., an ideal gas) will have three equal contributions for each particle corresponding to the three orthogonal direction of translation. On the other hand, a diatomic gas will have seven equal contributions, six equal contributions (three each) from the momenta of the two atoms and a seventh from the vibration between the two that can be represented by a potential. This result can be stated elegantly as:

$$\left\langle x_m \frac{\partial \mathcal{H}}{\partial x_n} \right\rangle = \delta_{mn} kT \tag{21}$$

where x_m may be q_m or p_m.

4.2.1 Example

Suppose we have a single one dimensional particle represented by the Hamiltonian of a "harmonic oscillator" given in (22)

$$\mathcal{H} = \frac{p^2}{2} + \frac{\omega^2}{2} q^2 \tag{22}$$

Now suppose this particle is under constant bombardment by numerous other small particles (in one dimension) and that momentum is transferred between the particles upon collision. Also, suppose that the probability of collision is equally likely to come from the right or the left so that the net momentum transfer (and net resulting force) is zero. Then

$$\begin{aligned} \langle p \rangle &= 0 \\ \langle q \rangle &= 0 \end{aligned}$$

From the Equipartition theorem, (21), we obtain,

$$\begin{aligned} \left\langle p \frac{\partial \mathcal{H}}{\partial p} \right\rangle &= kT && (23) \\ \text{So that} \qquad \left\langle p^2 \right\rangle &= kT \\ \text{And} \qquad \left\langle q \frac{\partial V(q)}{\partial q} \right\rangle &= kT \\ \left\langle q^2 \right\rangle &= kT/\omega^2 && (24) \end{aligned}$$

These are consistent with (20) which gives us the equilibrium distribution,

$$\begin{aligned} \rho(q,p) &= \frac{1}{2\pi kT} \exp\left(-\frac{\mathcal{H}(q,p)}{2kT} \right) \\ &= \frac{1}{2\pi kT} \exp\left(-\frac{p^2 + \omega^2 q^2}{2kT} \right) && (25) \end{aligned}$$

Thus for a quadratic potential, we can find that the joint distribution of net inventory and net flow for the system once it has achieved steady state (equilibrium) is a bivariate normal distribution with no correlation between q and p.

The implication here is that the average kinetic energy for the particle is proportional to the temperature that is established by the myriad of smaller particles. This implies that the average kinetic energy of both the particle in question and the smaller particles will be the same in equilibrium. To be useful in modeling a supply chain, we must relate kT to supply chain parameters representing the mean and variance of demand.

5 Modeling the Supply Chain

We now apply these concepts to a supply chain. We begin by considering models for demand. Then, we add potential terms to represent the control imposed.

It appears that of the many operations research models employed to model demand, the most common are renewal processes (in particular, Poisson processes, see, e.g., [6]), and Brownian motion. Both of these processes have what is known as "quadratic variation" in which the variance of the demand grows as a linear function of time (see § 10.2 for a complete discussion).

While the linear growth of the variance may commonly appear in supply chain literature, it is by no means the only relation found in many *physical* processes. Indeed, a free "quantum" particle (i.e., a particle with zero potential), does *not* exhibit quadratic variation. This is significant because quantum physics is a branch of physics in which randomness is intrinsic and fundamental. Instead, the variance of the location of a free quantum particle increases as t^2. On the other hand, there are many examples from statistical physics (including the original motion studied by Brown) that involve processes that exhibit quadratic variation.

We now postulate a Least Action formulation for supply chain demand using the above developments.

5.1 A Hamiltonian Model for Demand

We first add a random force to represent the fluctuations in demand. These fluctuations are similar to a set of many small particles continuously striking a larger particle in one dimension. However, in the presence of these many small particles there must also be some sort of *damping* process because, as Kubo [15] explains, both the random fluctuations and the damping are caused by the same physical process (i.e., the small particles striking the larger one). Thus, in the presence of damping, we will require some sort of "force," representing the demand of the system, pushing (or, "pulling" if the enthusiasts insist) on the particle representing the net-inventory. This force can be represented by a linear potential, $V(q) = -F_0 q$ and the Hamiltonian will be

$$\mathcal{H} = \frac{p^2}{2} - F_0 q \tag{26}$$

We will also need a damping term that results from a force that is proportional to the velocity of the particle acting in the opposite direction, $-\eta p$, where $1/\eta$ acts as a "time constant" indicating how quickly the system reaches the final value of net-flow.

To represent random fluctuations in demand, we add another force, $\beta F_r(t)$ where β is the "diffusion" coefficient and $F_r(t)$ is a stochastic process with mean zero and a variance that depends on t and is further specified below. Our equations of motion become,

$$\dot{p} = F_0 - \eta p + \beta F_r(t) \quad (27)$$
$$\dot{q} = p \quad (28)$$

Despite these non-conservative forces, we can solve the resulting equations of motion. However, we would not expect for the energy of the system to remain constant although, in steady state, it will achieve a fixed value, namely kT.

5.2 Specifying the Stochastic Process

Interestingly, if we set $F_0 = 0$, equation (27) becomes *Langevin's equation* that was first proposed in 1908 to describe the dynamics of *physical* Brownian motion [14]. Thus, it appears that Brownian motion arises naturally in the Hamiltonian formulation.

For Brownian motion to be an appropriate model we need only two conditions: that the path is continuous and that non-overlapping increments are probabilistically independent. The other conditions usually discussed (e.g., normality) can either be imposed or arise from these (see Harrison [9] and § 10.2).

However, there are fundamental problems with this formulation. Because Brownian motion represents discrete jumps at a point in time, the model implies infinite velocities that are not possible. Nonetheless, the model works very well and predicts results well within the experimental error.[4] The reason it works well is that whenever a "large" particle such as a pollen grain is suspended in a liquid, there are an incredible number of much smaller liquid molecules colliding with it each second. That the collisions are so numerous and are independent of each other results in normality via the Central Limit Theorem. For it to work well in a supply chain setting, we would need similar conditions—a large number of independent demands occurring during a typical time period. If the number of demands were less, we would need to increase the period size so that a significant number of demands (say, at least 10) occur in any period.

Equation (27) can be written as a SDE,

$$\begin{aligned} dp &= (F_0 - \eta p)\, dt + \beta dW \\ p(0) &= p_0 \end{aligned} \quad (29)$$

where W represents a standard Brownian motion process (also known as a "Wiener" process) and β is the diffusion coefficient. Given the way W is defined, it has effective "dimensions" of $\sqrt{t}$ so that β has dimensions of $p/\sqrt{t}$. The solution to the above is obtained by the usual methods recognizing βdW as the differential of an "external" force.

$$p(t) = p_0 e^{-\eta t} + \frac{F_0}{\eta}\left(1 - e^{-\eta t}\right) + \beta \int_0^t e^{-\eta(t-s)} dW \quad (30)$$

[4] Indeed, the approximate model for Brownian motion was used by Jean Perrin to determine the value of Avogadro's number, an accomplishment for which he won the Nobel Prize in 1926.

Again the last integral is a random variable whose expectation and variance may be found using the specific properties listed in § 10.1. The expected value is found using (62),

$$\langle p(t)\rangle = \langle p_0\rangle e^{-\eta t} + \frac{F_0}{\eta}\left(1 - e^{-\eta t}\right) \tag{31}$$

so that the terminal velocity is F_0/η. This terminal velocity would correspond to the mean demand, λ, so that $\lambda = F_0/\eta$.

The variance of p is found using (63),

$$\begin{aligned}
\mathrm{var}(p(t)) &= \mathrm{var}(p_0)e^{-2\eta t} + E\left[\left(\beta\int_0^t e^{-\eta(t-s)}dW\right)^2\right] \\
&= \mathrm{var}(p_0)e^{-2\eta t} + \beta^2\int_0^t e^{-2\eta(t-s)}ds \\
&= \mathrm{var}(p_0)e^{-2\eta t} + \frac{\beta^2}{2\eta}\left(1 - e^{-2\eta t}\right)
\end{aligned} \tag{32}$$

so that the limiting variance is $\beta^2/(2\eta)$. In a controlled situation where the expected value of the momentum is zero, the limiting variance is also $\langle p^2\rangle$. Then, from the Equipartition Theorem (see e.g., [12]), we know that for any controlled PI system,

$$\langle p^2\rangle = \frac{\beta^2}{2\eta} = kT \tag{33}$$

Note that the variance of the steady state momentum (i.e., net flow) will *always* be equal to $\beta^2/(2\eta)$, regardless of the potential.

The position can be found by noting

$$q(t) = q_0 + \int_0^t p(s)ds$$

so that

$$\begin{aligned}
q(t) &= q_0 + \int_0^t \left(V_0 e^{-\eta s} + \frac{F_0}{\eta}\left(1 - e^{-\eta s}\right) + \beta\int_0^s e^{-\eta(s-x)}dW\right)ds \\
&= q_0 + \frac{V_0}{\eta}(1 - e^{-\eta t}) + \frac{F_0}{\eta}\left(t - \frac{1}{\eta}\left(1 - e^{-\eta t}\right)\right) + \frac{\beta}{\eta}\int_0^t \left(1 - e^{-\eta s}\right)dW
\end{aligned} \tag{34}$$

where q_0 and V_0 could be random variables and the last integral is again a random variable. The expected value is

$$\langle q\rangle = \langle q_0\rangle + \frac{\langle V_0\rangle}{\eta}(1 - e^{-\eta t}) + \frac{F_0}{\eta}\left(t - \frac{1}{\eta}\left(1 - e^{-\eta t}\right)\right) \tag{35}$$

When q_0 and V_0 are constants, the variance of q is obtained by applying (63) to the Itô integral in 34.

$$\mathrm{var}(q(t)) = \frac{\beta^2}{\eta^2}\left(t - \frac{2}{\eta}\left(1 - e^{-\eta t}\right) + \frac{1}{2\eta}\left(1 - e^{-2\eta t}\right)\right) \tag{36}$$

Note that as t goes to infinity, the variance becomes a linear function of time that can be related to the demand variance, viz.

$$\lim_{t\to\infty}\frac{\mathrm{var}(q(t))}{t} = \frac{\beta^2}{\eta^2} = \psi^2\lambda \tag{37}$$

6 A General Model for Production-Inventory Systems

We are now ready to offer a general model for any production-inventory system. Consider a multi-product production-inventory system with net-flow of $\mathbf{p}$ and net-inventory of $\mathbf{q}$ and controlled by a potential, $V(\mathbf{q})$.

Equation (29) demonstrates how a force can create a net flow that is demand only. This example represents an uncontrolled system in which the expectation of the net-flow does not change. The potential to generate such a force is $V(q) = F_0 q = \eta\lambda q$. We can also create state-dependent potentials that can be used to control inventory.

6.1 The General Equation

Using such potentials, the general equations for a PI system can be derived using Hamilton's equations combined with the elements of a Brownian motion.

Definition. *General equations of a production-inventory system.*
The following equations describe the behavior of a general multi-product production-inventory system with a Hamiltonian of $\mathcal{H}$, a damping coefficient of η, and a diffusion coefficient of β.

$$\begin{aligned} d\mathbf{p} &= (\{\mathbf{p}, \mathcal{H}\} - \eta\mathbf{p})\, dt + \beta dW \\ d\mathbf{q} &= \{\mathbf{q}, \mathcal{H}\} dt \\ \mathbf{p}(0) &= p_0 \\ \mathbf{q}(0) &= q_0 \end{aligned} \tag{38}$$

For a single particle we can expand the Poisson brackets and substitute $F(q) = -\partial\mathcal{H}/\partial q$ yielding,

$$\begin{aligned} dp &= (F(q) - \eta p)\, dt + \beta dW \\ dq &= p\, dt \\ p(0) &= p_0 \\ q(0) &= q_0 \end{aligned} \tag{39}$$

With this we can immediately write the steady state distribution of the net flow and net inventory as,

$$\rho(q, p) = C \exp\left(-\frac{\mathcal{H}(q, p)}{kT}\right) \tag{40}$$

where

$$C = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(q, p) dq dp\right)^{-1}$$

and $kT = \beta^2/(2\eta)$. Equations (38) and (39) can be used to determine the transient behavior of such systems while (40) can be used to determine probabilities along with the expected values and variances of p and q. For more complex potentials, the equations of (38) provide an easy means to simulate such a system (see, e.g., [5] for an excellent treatment of the numerical integration of SDE's).

6.2 Example: Damped harmonic oscillator

The harmonic oscillator is a classic physics problem. In our setting the corrective force is proportional to the difference between the net-inventory and its target. The quadratic potential and linear force in this type of control is reminiscent to the control proposed in the classic paper by Holt, et al. [11]. The Hamiltonian for the undamped harmonic oscillator is

$$\mathcal{H} = \frac{p^2}{2} + \frac{\omega^2}{2} q^2 \tag{41}$$

The previous example involved a first order differential equation involving velocity but could have been written as a second order differential equation in terms of q as,

$$\ddot{q} + \eta \dot{q} = F(q) + \beta W(t)$$

where $F(q) = -\partial V(q)/\partial q$ and $\beta W(t)$ is an external Brownian motion force. A simpler differential equation with a corrective force that is proportional to the displacement from a target (here, without loss of generality, set to zero), is given by

$$\ddot{q} + \eta \dot{q} + \omega^2 q = \beta W(t) \tag{42}$$

Such an ordinary second order differential equation with constant coefficients describes the motion of a "damped harmonic oscillator" where ω is a "natural frequency." Note that the restoring force is $-\omega^2 q$.

The behavior of the system depends greatly on the natural frequency of the system. If $\omega < \eta/2$ the system will be "over-damped" and will take longer to reach equilibrium. On the other hand, if $\omega > \eta/2$, the system is said to be "under-damped" and will oscillate above and below the target before reaching equilibrium. However, if $\omega = \eta/2$, the system is "critically-damped" and will reach equilibrium in minimum time without going below zero. Figure 1 demonstrates these cases while figure 2 provides an example of a sample path that is critically damped. See § 10.4 for a complete derivation of the solutions and the steady state results. The results for an "underdamped" system are given below. We first define two constants,a and b such that

$$a \pm ib = \frac{-\eta \pm i\sqrt{4\omega^2 - \eta^2}}{2}$$

then the expectations and variances can be computed using Itô calculus,

$$\begin{aligned}
\langle q(t) \rangle &= e^{at}\left(q_0 \cos bt + \frac{V_0 - aq_0}{b} \sin bt\right) && (43)\\
\mathrm{var}(q(t)) &= \frac{\beta^2}{4ab^2(a^2+b^2)}\left[e^{2at}\left(a^2 + b^2 - a(a\cos 2bt + b \sin 2bt)\right) - b^2\right] && (44)\\
\lim_{t\to\infty} \mathrm{var}(q(t)) &= \frac{\beta^2}{2\eta\omega^2} && (45)\\
\left\langle p^2(t) \right\rangle &= \frac{\beta^2}{4ab^2}\left[e^{2at}\left(a^2 + b^2 - a(a\cos 2bt + b\sin 2bt)\right) - b^2\right] && (46)\\
\lim_{t\to\infty} \left\langle p^2(t) \right\rangle &= -\frac{\beta^2}{4a} = \frac{\beta^2}{2\eta} && (47)
\end{aligned}$$

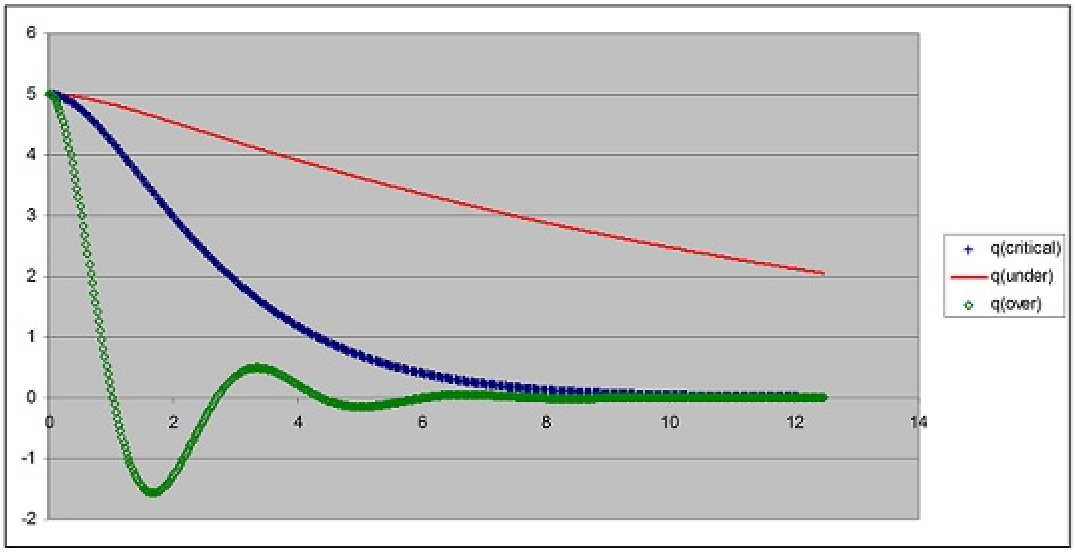

Figure 1: Comparison of different levels of damping.

We set our "target" net inventory to be zero to make the calculations simpler but we could have made it any value. That the average net inventory goes to zero in (43) means that the system converges to the target value, in expectation. We could have more easily obtained the steady state variances given by (45) and (47) directly from

$$\begin{aligned}\rho(q,p) &= C\exp\left(-\frac{\mathcal{H}(q,p)}{kT}\right)\\ &= \frac{\omega}{2\pi kT}\exp\left(-\frac{p^2+\omega^2q^2}{2kT}\right)\end{aligned}$$

and identifying $kT = \langle p^2\rangle = \beta^2/(2\eta)$. This result indicates that the steady state mean values of both q and p are zero and their variances are,

$$\begin{aligned}\lim_{t\to\infty}\mathrm{var}(q(t)) &= \frac{\beta^2}{2\eta\omega^2} = \frac{kT}{\omega^2}\\ \lim_{t\to\infty}\mathrm{var}(p^2(t)) &= -\frac{\beta^2}{4a} = \frac{\beta^2}{2\eta} = kT\end{aligned}$$

7 Implementing the Hamiltonian Control

The above analysis assumes that the system is controlled continuously. This would mean constantly adjusting production in an attempt to keep the net-inventory on target. Of course, this would be impractical and so changes are made in discrete time increments.

If we naively attempt to implement such a strategy, the system could become unstable as well as unpredictable. Fortunately, this problem is not unlike that of numerically integrating a stochastic differential equation where the issues of stability and predictability

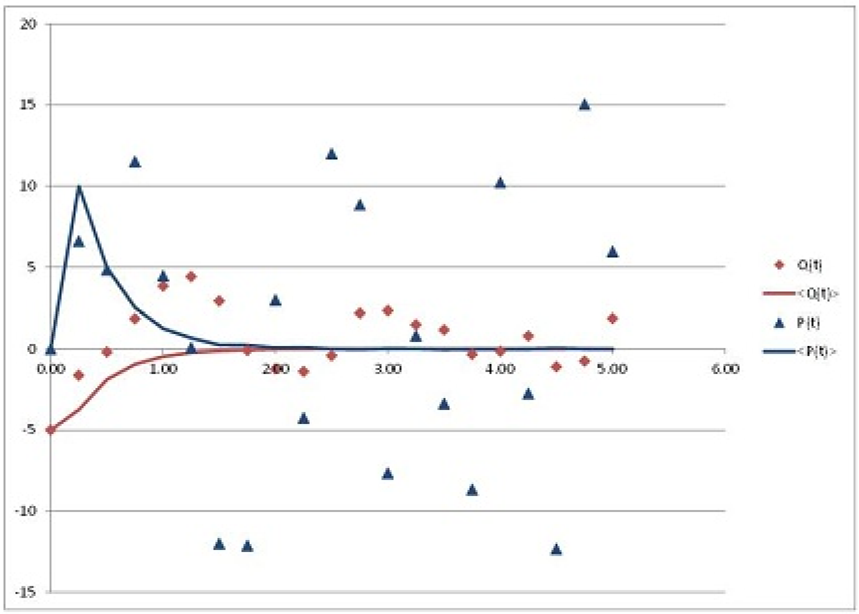

Figure 2: Sample path of critically damped system.

are paramount. Thus, using such numerical methods to periodically recalculate production rates should be both practical and predictable.

7.1 Control Methods from Numerical Integration of SDE

An easy method is the "modified leap-frog" method by Mannella [17] that can be implemented in our notation as,

$$\begin{aligned}
\hat{Q} &= Q_n + \frac{1}{2}P_n\Delta t \\
P_{n+1} &= c_2\left(c_1 P_n + f(\hat{Q})\Delta t + \beta\Delta W_n\right) \\
Q_{n+1} &= \hat{Q} + \frac{1}{2}P_{n+1}\Delta t
\end{aligned}$$

where $c_1 = 1 - \frac{1}{2}\eta\Delta t$, $c_2 = \left(1 + \frac{1}{2}\eta\Delta t\right)^{-1}$, and $f(\hat{Q}) = -\left.\frac{\partial V(q)}{\partial q}\right|_{\hat{Q}}$. Note that ΔW_n is a normal random variable with zero mean and variance of Δt. Applying this method to the damped harmonic oscillator results in the correct the steady state variance of q while the error in var(p) is independent of η.

The harmonic oscillator is important because, not only does it result in the minimum uncertainty, but any system in equilibrium will behave, as a first order approximation, like a harmonic oscillator. Thus, the behavior of a harmonic oscillator in equilibrium will be a reasonable approximation of the behavior of almost any potential for small perturbations. So, let $f(q) = -\omega^2 q$ and define the average demand, D_n and production, X_n as,

$$D_n = c_2\left(\eta\theta\Delta t - \beta W_n\right) \tag{48}$$

$$X_n = c_2\left(-\omega^2\hat{Q} + \eta\theta\right)\Delta t \tag{49}$$

where θ and β will be selected to match the demand parameters. Note that D_n contains all of the randomness and X_n depends only on previous performance. Then,

$$P_{n+1} = c_1 c_2 P_n + X_n - D_n \tag{50}$$

The average demand determines θ

$$\begin{aligned}
\langle D\rangle = \lambda &= c_2\eta\theta\Delta t \\
\lambda &= \frac{\eta\theta\Delta t}{1 + \frac{1}{2}\eta\Delta t}
\end{aligned}$$

so that,

$$\theta = \frac{\lambda}{\eta\Delta t}\left(1 + \frac{1}{2}\eta\Delta t\right) \tag{51}$$

Likewise,

$$\begin{aligned}
\mathrm{var}(D\Delta t) = c_2^2\beta^2(\Delta t)^3 &= \psi^2\lambda\Delta t \\
\frac{\beta^2(\Delta t)^2}{\left(1 + \frac{1}{2}\eta\Delta t\right)^2} &= \psi^2\lambda
\end{aligned}$$

or,

$$\beta^2 = \frac{\psi^2\lambda\left(1 + \frac{1}{2}\eta\Delta t\right)^2}{(\Delta t)^2} \tag{52}$$

7.2 Variances in the Discrete Time Model

The above method is guaranteed to yield the same variance for Q as the theoretical continuous case

$$\mathrm{var}(Q) = \frac{\beta^2}{2\eta\omega^2} = \frac{kT}{\omega^2} \tag{53}$$

The variance of the production rate, X, is

$$\begin{aligned}
\mathrm{var}(X) &= \mathrm{var}\left(c_2(-\omega^2\hat{Q})\Delta t\right) \\
&= c_2^2\omega^4(\Delta t)^2\mathrm{var}(\hat{Q}) \\
&= c_2^2\omega^4(\Delta t)^2\mathrm{var}(Q + \tfrac{1}{2}P\Delta t) \\
&= c_2^2\omega^2(\Delta t)^2 kT\left(1 + \frac{\omega^2(\Delta t)^2}{4}\right)
\end{aligned} \tag{54}$$

Then the uncertainty relation will be,

$$\sigma_X\sigma_Q = c_2 kT\left(1 + \frac{\omega^2(\Delta t)^2}{4}\right)^{1/2}\Delta t \tag{55}$$

7.3 Relation Between the Hamiltonian and the Basic Production-Inventory Models

It would appear that the Hamiltonian formulation is completely different from the base stock model. The base stock is a first order SDE while the Hamiltonian model is a second order one. However, when we convert to discrete time, we find that the base stock model is a special case of the Hamiltonian model.

Note that if $\eta\Delta t = 2$ then $c_1 = 0$ and $c_2 = 1/2$ and

$$\theta = \frac{\lambda}{2}(1+1) = \lambda \tag{56}$$

Likewise,

$$\begin{aligned}
\mathrm{var}(D\Delta t) = c_2^2\beta^2(\Delta t)^3 &= \psi^2\lambda\Delta t \\
\frac{\beta^2(\Delta t)^2}{2^2} &= \psi^2\lambda \\
\frac{\beta^2}{\eta^2} &= \psi^2\lambda
\end{aligned}$$

or,

$$\begin{aligned}
\beta^2 &= \frac{\psi^2\lambda\left(1+\frac{1}{2}\eta\Delta t\right)^2}{(\Delta t)^2} \\
&= \frac{\psi^2\lambda\,(2)^2}{(\Delta t)^2} \\
&= \psi^2\lambda\eta^2
\end{aligned}$$

Moreover, it is easy to show that the equation for $\hat{Q}$ is *exactly* the equation for the base stock model. First, note that the discrete form of the base stock model can be written as

$$Q_{n+1} = Q_n - gQ_n + \psi\sqrt{\lambda\Delta t}Z_n$$

$$\begin{aligned}
\hat{Q}_n &= Q_n + \tfrac{1}{2}P_n\Delta t \\
P_{n+1} &= \frac{1}{2}\left(-\omega^2\hat{Q}_n\Delta t + \beta\sqrt{\Delta t}\,Z\right) \\
Q_{n+1} &= \hat{Q}_n + \tfrac{1}{2}P_{n+1}\Delta t \\
\hat{Q}_{n+1} &= Q_{n+1} + \frac{1}{2}P_{n+1}\Delta t \\
&= \hat{Q}_n + P_{n+1}\Delta t \\
\hat{Q}_{n+1} &= \hat{Q}_n + \frac{1}{2}\left(-\omega^2\hat{Q}_n\Delta t + \beta\sqrt{\Delta t}\,Z_n\right)\Delta t \\
&= \hat{Q}_n - g\hat{Q}_n + \psi\sqrt{\lambda\Delta t}Z_n
\end{aligned} \tag{57}$$

where

$$\begin{aligned}
\tfrac{1}{2}\omega^2\Delta t &= g \\
\tfrac{1}{2}\beta\Delta tef &= \frac{\beta}{\eta} = \psi\sqrt{\lambda}
\end{aligned}$$

Since $\hat{Q}_n = Q_n + P_n\Delta t/2$ and since, in equilibrium, Q and P are independent, normally distributed random variables, the variance of $\hat{Q}$ will always be greater than the variance of Q. Consequently, the base stock model will always have greater variance than that for the equivalent damped harmonic oscillator.

7.4 Comparison of the Variances of the Discrete Systems

When the two models are equivalent, the variance of net inventory for the discrete time DHO can be written in terms of $\psi^2\lambda$ and g by noting, $\eta = 2/\Delta t$ and $\omega^2 = 2g/\Delta t$, viz.

$$\text{var}(Q) = \frac{\beta^2}{2\eta\omega^2} = \frac{\psi^2\lambda}{\omega^2\Delta t} = \frac{\psi^2\lambda}{2g}$$

We can use these relations and (54) to obtain the variance of production.

$$\text{var}(X) = \frac{\psi^2\lambda}{2}g\left(1 + g\frac{\Delta t}{2}\right) \tag{58}$$

so that,

$$\sigma_X\sigma_Q = \frac{\psi^2\lambda}{2}\left(1 + g\frac{\Delta t}{2}\right)^{1/2} \tag{59}$$

We can also obtain the variances of the base stock model from

$$\begin{aligned}
\text{var}(Q_b) &= \text{var}(\hat{Q}) \\
&= \frac{\psi^2\lambda}{2}\frac{\eta}{\omega^2}\left(1 + \frac{\omega^2(\Delta t)^2}{4}\right) \\
&= \frac{\psi^2\lambda}{2g}\left(1 + g\frac{\Delta t}{2}\right)
\end{aligned}$$

Likewise, the variance of X_b,

$$\begin{aligned}\mathrm{var}(X_b) &= g^2\mathrm{var}(Q_b) \\ &= \frac{\psi^2\lambda}{2}g\left(1+g\frac{\Delta t}{2}\right)\end{aligned}$$

Thus,

$$\sigma_{X_b}\sigma_{Q_b} = \frac{\psi^2\lambda}{2}\left(1+g\frac{\Delta t}{2}\right) \tag{60}$$

Note that, for positive values of g, (59) will always be less than (60).

7.5 Example Simulation of Control

Now suppose we have Poisson demand with an average of 16 units per period so that $\psi^2 = 1$ and $\lambda = 16$ day^{-1} and we use the basic control with $g = 1$. To match the traditional model we set $\eta\Delta t = 2$. This implies $\theta = \lambda = 16$ day^{-1}. For stability we choose $\Delta t = 1/3$ so that $\eta = 6$. Then, for the DHO,

$$\begin{aligned}\mathrm{var}(Q_D) &= \frac{\psi^2\lambda}{2g} = 8.000 \\ \mathrm{var}(X_D) &= g\frac{\psi^2\lambda}{2}\left(1+g\frac{\Delta t}{2}\right) = 9.333\end{aligned}$$

so that,

$$\sigma_{X_D}\sigma_{Q_D} = 8.641$$

And for the BSM,

$$\begin{aligned}\mathrm{var}(Q_B) &= \frac{\psi^2\lambda}{2g}\left(1+g\frac{\Delta t}{2}\right) = 9.333 \\ \mathrm{var}(X_B) &= g\frac{\psi^2\lambda}{2}\left(1+g\frac{\Delta t}{2}\right) = 9.333\end{aligned}$$

and

$$\sigma_{Q_B}\sigma_{X_B} = 9.333$$

The simulation involves 200,000 replicates over 18 periods of 1/3 each. Since the system begins in steady state (i.e., $Q(0) = 0, P(0) = 0$), there is no transient period. The results shown at the top of Table 1 confirm the fact that the Mannella algorithm converges to the correct value of the variance of Q and is very close to the variance of X as well. Regardless, the simulation is actually the "correct" value when we are controlling a real system at points in time (as opposed to a continuous control). Thus, the base stock model has roughly 8-9% more variability than the damped harmonic oscillator model (comparing the products of the simulated standard deviations).

Can we do better? It appears so. If we set $\eta = 10$ and run the DHO model. The variance of both measures is predicted to drop to 8.5 for Q and 5.6 for X yielding a product of standard deviations of 6.9 as compared to 9.3 for the base stock model. The simulation generated a value of 7.0 for the same product. Thus, the improvement over the base stock model is around 35%.

Matched DHO and BSM			
Measure	Simulation	Theoretical	% Error
DHO var(Q)	7.999	8.000	-0.02%
DHO var(X)	9.628	9.333	3.15%
BSM var(Q)	9.628	9.333	3.15%
BSM var(X)	9.578	9.333	2.63%
DHO $\sigma_Q \sigma_X$	8.775	8.641	1.56%
BSM $\sigma_Q \sigma_X$	9.603	9.333	2.89%
$\sigma_Q \sigma_X$ difference	8.01%	9.43%	
Enhanced DHO and BSM			
Measure	Simulation	Theoretical	% Error
DHO var(Q)	8.536	8.533	0.04%
DHO var(X)	5.774	5.600	3.11%
BSM var(Q)	9.628	9.333	3.15%
BSM var(X)	9.578	9.333	2.63%
DHO $\sigma_Q \sigma_X$	7.021	6.913	1.56%
BSM $\sigma_Q \sigma_X$	9.603	9.333	2.89%
$\sigma_Q \sigma_X$ difference	36.78%	35.02%	

Table 1: Simulation results for the BSM and equivalent DHO.

8 Conclusions

We have seen that the Hamiltonian formulation of the Least Action model can be applied to the supply chain with interesting results. It appears that the traditional method of controlling production to hit an inventory target is a special case of this Hamiltonian model. As such, the Hamiltonian model cannot perform worse and, indeed, can perform as a more efficient production control method.

Adding randomness to the Hamiltonian model quite "naturally" results in a Brownian motion. Such models have seen a great deal of application in the supply chain but were typically postulated as a reasonable a priori model and not derived from basic principles. Our development here provides additional justification for such models.

This exploration has also suggested new ways to control production and inventory. Whereas most of previous inventory models have been limited in the number of possible probability distributions, the possibility of using a variety of "potential" functions as controls suggests an almost limitless number of ways to provide control. Moreover, the distribution of net-inventory of any such control is easily constructed from the potential used,

$$\rho(q) = C \exp\left(-\frac{2\eta V(q)}{\beta^2}\right)$$

where C is used to normalize the distribution.

Finally, the proposed model links the three "buffers" of inventory, time, and capacity in a "canonical" way. Although the link between the time and inventory buffer was already well understood (with backorder time proportional to negative net-inventory, see e.g., [26]), the link between these two and capacity has been studied heretofore only for special cases. We have now seen a direct link between production and net-inventory as these two are "canonical conjugate" variables and have an associated "uncertainty principle" that is directly related to the idea of buffers mitigating variability.

$$\sigma_x \sigma_q \geq \frac{\psi^2 \lambda}{2}$$

8.1 Future Work

We have only begun to scratch the surface of the Least Action model. Other areas of exploration include:

Optimization of the supply chain. The uncertainty relation shows that if we want tighter control of inventory, we must accommodate a larger variance in production and therefore have a larger capacity "buffer." Conversely, if we cannot afford the extra capacity, we will have to live with either larger inventories or worse customer service. This relation provides a quantitative means to optimize the supply chain considering the cost of capacity, service (time), and inventory in a way to minimize total life cycle costs. This should be explored in future work.

How is the Hamiltonian model related to system dynamics models such as those developed by Forrester [7] and more recently explored by, Sterman [24]? Both the Hamiltonian model and system dynamics involve differential equations and damping. Is the Hamiltonian control simply a coincidental example of a systems dynamics model? If so, is there an underlying structure?

Conservation and Symmetry. The relation between symmetry and conservation laws runs deep in physics. Noether's theorem (see § 10.3 for a short proof) suggests that for every symmetry in the Hamiltonian there must be some conservation law. An interesting question is whether there are similar symmetries and conservation laws in supply chain dynamics. If not, why not? If so, what are they?

The nature of randomness in a supply chain. As we mentioned above, the branch of physics that deals with intrinsic randomness is quantum physics. Moreover, quantum physics does not exhibit quadratic variation but it appears that supply chain processes do. Nonetheless, our derivation was based on "statistical physics" which also exhibits quadratic variation. But statistical physics is a misnomer because it is not random at all but involves an incredibly large number of elementary states. So the question is whether the supply chain is intrinsically random or simply complex and unknown.

Other Applications. Can the Least Action Principle be applied to other areas of operations management?

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10 Appendices

10.1 Properties of Itô Integrals

The following properties make it easy to compute expectations of the solutions to stochastic differential equations (see, e.g., [19]).

Consider the following differential equation,

$$\dot{p} = f(p) + g(p)\frac{dW}{dt}$$

where W is a Wiener process. The implication here is that dW/dt is "white noise" (i.e., a random process with a constant power spectral density). We can rewrite this equation along with its initial condition as a *stochastic differential equation*

$$\begin{aligned} dp(t) &= f(p)dt + g(p)dW \\ p(0) &= p_0 \end{aligned}$$

The above has $p(t)$ as its solution provided

$$p(t) = p_0 + \int_0^t f(s)ds + \int_0^t g(s)dW$$

where the last term is an *Itô integral* and is a random variable.

Various moments and other properties of the Itô integral can be computed for all constants $a, b \in \Re$ and for all $G, H \in L^2[0, T]$ (an L^2 Lebesgue space or, equivalently, a Hilbert space), viz.

$$\int_0^T (aG + bH)dW = a\int_0^T GdW + b\int_0^T HdW \tag{61}$$

$$\left\langle \int_0^T GdW \right\rangle = 0 \tag{62}$$

$$\left\langle \left(\int_0^T GdW \right)^2 \right\rangle = \left\langle \int_0^T G^2 dt \right\rangle \tag{63}$$

$$\left\langle \int_0^T GdW \int_0^T HdW \right\rangle = \left\langle \int_0^T GHdt \right\rangle \tag{64}$$

10.2 Demand Models

Commonly used models of demand include renewal processes (especially Poisson processes) and Brownian Motion models.

First, consider a renewal process where X_i the time between renewals and is iid with a mean of μ, a variance of σ^2, and a third central moment of μ_3 for all i. Define, S_n to be the time of the n^{th} renewal,

$$S_n = \sum_{i=1}^{n} X_i$$

Now let N_t be the number of renewals in $(0, t)$. Then

$$N_t < r \text{ if and only if } S_r > t$$

From [6], the limiting mean and variance of N_t in time are found to be

$$\langle N_t \rangle = \frac{t}{\mu} + \frac{1}{2}\frac{(\sigma^2 - \mu^2)}{\mu^2} + o(1) \tag{65}$$

$$\text{var}(N_t) = \frac{\sigma^2 t}{\mu^3} + \left(\frac{1}{12} + \frac{5\sigma^4}{4\mu^4} + \frac{\mu_3}{3\mu^3}\right) + o(t^{-1}) \tag{66}$$

Note the mean and variance of demand both increase linearly in time.

Unlike a renewal process, a Brownian motion can be used to model demand, production, or net-flow. A standard Brownian motion (also called a Wiener Process) is defined as:

Definition. *Standard Brownian Motion*
A stochastic process, $\{W_t : 0 \leq t < \infty\}$ is a standard Brownian motion if

1. $W_0 = 0$, almost surely.
2. It has continuous sample paths.
3. It has independent, normally distributed increments.

Harrison [9] reports a more general result,

Theorem. *If Y is a continuous process with stationary independent increments, then Y is a Brownian motion.*

This means that we need not require the incremental distribution to be normal and that the normality follows from the other two requirements. Thus, for demand to be well modeled by a Brownian motion we require only independent time increments and for the level of demand to be high enough so that the set of integers appear "continuous."

Harrison presents a third result known as "quadratic variation" that implies that the variance of a Brownian motion increases linearly in time. So, Brownian motion, like a renewal counting process has a mean and a variance that both increase linearly in time.

$$\lim_{t\to\infty} \langle N_t - \lambda t \rangle = 0 \tag{67}$$

$$\lim_{t\to\infty} \text{var}(N_t) - \psi^2 \lambda t = 0 \tag{68}$$

where $\lambda = 1/\mu$ and $\psi^2 = \sigma^2/\mu^2$.

10.3 Noether's Theorem

Noether's theorem shows that for any differential symmetry of the action of a physical system, there is a corresponding conservation law. We can now provide a simple proof for a one dimensional system but the logic can be extended to as many dimensions as necessary. Moreover, although we describe the coordinate as q this need not be a translational coordinate. Angular coordinates and any other generalized coordinate along with the associated momentum work equally well.

Consider a particle whose motion is determined by the Lagrangian $L(q, \dot{q})$ with momentum defined by,

$$p = \frac{\partial L}{\partial \dot{q}}$$

Now suppose this Lagrangian has a time-independent *symmetry* defined by a one-parameter family of transformations: $q \to q(s)$ such that,

$$\frac{d}{ds} L(q(s), \dot{q}(s)) = 0.$$

We now show that quantity $p\frac{dq(s)}{ds}$ is constant in time (i.e., a conserved quantity). First, by the chain rule and substituting in the definition of momentum,

$$\frac{d}{dt}\left(p\frac{dq(s)}{ds}\right) = \dot{p}\frac{dq(s)}{ds} + \frac{\partial L}{\partial \dot{q}}\frac{d\dot{q}(s)}{ds}$$

But we know that

$$\dot{p} = -\frac{\partial V(q)}{\partial q} = \frac{\partial L}{\partial q}$$

so that,

$$\frac{d}{dt}\left(p\frac{dq(s)}{ds}\right) = \frac{\partial L}{\partial q}\frac{dq(s)}{ds} + \frac{dL}{d\dot{q}}\frac{d\dot{q}(s)}{ds}$$

But the right hand side is simply,

$$\frac{d}{ds} L(q(s), \dot{q}(s)) = 0$$

so that $p\frac{dq(s)}{ds}$ is a conserved quantity.

10.3.1 Examples

Allowing q to be a translational coordinate shows that spatial symmetry implies conservation of linear momentum. If q is an angular coordinate about a particular axis, then rotational symmetry about that axis implies conservation of angular momentum about the same axis.

We have already seen symmetry in time exhibited by

$$\frac{dg}{dt} = \frac{\partial g}{\partial t} - \{\mathcal{H}, g\}$$

which means that g is conserved if it is not explicitly a function of time and if it commutes with the Hamiltonian. If the Hamiltonian is not an explicit function of time and since the Hamiltonian commutes with itself, it represents a conserved quantity, namely energy.

10.3.2 Discrete Symmetries

It turns out that all conservation laws in nature are related to symmetries by Noether's theorem. For instance, conservation of charge results in "gauge symmetry." There are also discrete symmetries that cannot be represented by continuous parameter such as "parity" (i.e., reflection). Such symmetries do not give rise to conservation laws using classical mechanics but they do using quantum mechanics.

10.4 The Damped Harmonic Oscillator

The differential equation with constant coefficients for q, $\dot{q}$, and $\ddot{q}$ describes a damped harmonic oscillator. Examples include a mass on a spring with friction, a pendulum in a fluid, acoustical systems, and a circuit involving a resistor, an inductor, and a capacitor. The basic mechanics of a harmonic oscillators are important because any particle subject to a force in a stable equilibrium acts like a harmonic oscillator for small perturbations.

The oscillator may or many not be "driven" by an external force. The example we consider here is driven by a Brownian motion. The general equation is

$$\ddot{q} + \eta\dot{q} + \omega^2 q = B(t) \tag{69}$$

Here ω^2 is a "natural frequency" and, in this case, determines the restoring force, $-\omega^2 q$.

There are three possible solutions to the homogeneous equation (i.e., when $B(t) = 0$) depending on the parameters η and ω^2.

Case 1: Over damped. $\eta^2 > 4\omega^2$ Let,

$$a \pm b = \frac{-\eta \pm \sqrt{\eta^2 - 4\omega^2}}{2}$$

and the solution is

$$q(t) = \frac{V_0 - (a-b)q_0}{2b} e^{(a+b)t} - \frac{V_0 - (a+b)q_0}{2b} e^{(a-b)t}$$

Case 2: Critically damped. $\eta^2 = 4\omega^2$ Let $a = -\eta/2$ and the solution will be

$$q(t) = (V_0 + q_0(1-a))e^{at}$$

Case 3: Under damped. $\eta^2 < 4\omega^2$ Let,

$$a \pm ib = \frac{-\eta \pm i\sqrt{4\omega^2 - \eta^2}}{2}$$

and the solution will be

$$q(t) = e^{at}\left(q_0 \cos bt + \frac{V_0 - aq_0}{b} \sin bt\right)$$

We can rewrite equation (42) as a stochastic differential equation,

$$\begin{aligned} dp &= \left(-\omega^2 q - \eta p\right) + \beta dW \qquad (70) \\ dq &= p\, dt \\ q(0) &= q_0 \\ p(0) &= p_0 \end{aligned}$$

This system can be solved using the properties of Itô integrals along with a theorem from ordinary differential equations regarding the "particular integral" (see § 10.5 for the details). Applying this to the under damped case yields the solution to the stochastic differential equation (70)

$$q(t) = e^{at}\left(q_0 \cos bt + \frac{V_0 - aq_0}{b} \sin bt\right) + \frac{\beta}{b}\int_0^t e^{a(t-s)} \sin(b(t-s))dW \qquad (71)$$

By now it is clear that the expected value of q will be

$$\langle q(t)\rangle = e^{at}\left(q_0 \cos bt + \frac{V_0 - aq_0}{b} \sin bt\right) \qquad (72)$$

that vanishes as $t \to \infty$. The variance of q can be obtained from (71) and properties of Itô integrals. Rewrite (71) as

$$q = y + I$$

where y contains the constant terms and I is the Itô integral. Then the variance will be

$$\begin{aligned} \operatorname{var}(q(t)) &= y^2 + 2y\langle I\rangle + \langle I^2\rangle - y^2 = \langle I^2\rangle \\ &= \left\langle \left(\frac{\beta}{b}\int_0^t e^{a(t-s)} \sin(b(t-s))dW\right)^2 \right\rangle \\ &= \frac{\beta}{b}\int_0^t e^{2a(t-s)} \sin^2(b(t-s))ds \\ &= \frac{\beta^2}{4ab^2(a^2+b^2)}\left[e^{2at}\left(a^2 + b^2 - a(a\cos 2bt + b\sin 2bt)\right) - b^2\right] \qquad (73) \end{aligned}$$

Note the limit

$$\lim_{t\to\infty} \operatorname{var}(q(t)) = \frac{\beta^2}{2\eta\omega^2} \qquad (74)$$

To obtain $p(t)$ first rewrite equation (71) for $q(t)$ as

$$q(t) = x(t) + \beta\int_0^t y(t-s)dW$$

so that,

$$\begin{aligned} p(t) &= \frac{d}{dt}q(t) = \frac{d}{dt}\left(x(t) + \beta\int_0^t y(t-s)dW\right) \\ &= \dot{x}(t) + \beta\int_0^t \dot{y}(t-s)dW \qquad (75) \end{aligned}$$

The variance will be

$$
\begin{aligned}
\mathrm{var}(p(t)) = \langle p^2 \rangle &= \dot{x}^2(t) + 2\dot{x}(t)\left\langle \beta \int_0^t \dot{y}(t-s)dW \right\rangle \\
&\quad + \left\langle \left(\beta \int_0^t \dot{y}(t-s)dW \right)^2 \right\rangle - \dot{x}^2(t) \\
&= \left\langle \left(\beta \int_0^t \dot{y}(t-s)dW \right)^2 \right\rangle \\
&= \beta^2 \int_0^t \dot{y}^2(t-s)ds \\
&= \frac{\beta^2}{b^2} \int_0^t \left(e^{a(t-s)} \left[a \sin(b(t-s)) + b \cos(b(t-s)) \right] \right)^2 ds \\
&= \frac{\beta^2}{4ab^2} \left[e^{2at} \left(a^2 + b^2 - a(a \cos 2bt + b \sin 2bt) \right) - b^2 \right] \qquad (76)
\end{aligned}
$$

Note the similarity in (37) and (76). In the limit, the variance becomes

$$
\lim_{t \to \infty} \mathrm{var}(p(t)) = -\frac{\beta^2}{4a} = \frac{\beta^2}{2\eta} \qquad (77)
$$

Which is as expected.

10.5 The Particular Integral Theorem

The Particular Integral Theorem provides a means to solve non-homogeneous differential equations. For a proof, see e.g. [3].

Let $v(x)$ be the solution of the reduced equation,

$$
\ddot{y} + \eta \dot{y} + \omega^2 y = 0
$$

which satisfies the conditions,

$$
v(0) = 0, \quad \dot{v}(0) = 1
$$

Then the solution $y_1(x)$ of the complete equation,

$$
\ddot{y} + \eta \dot{y} + \omega^2 y = \varphi(x)
$$

which satisfies the conditions,

$$
y_1(0) = 0, \quad y_1'(0) = 0
$$

is given by

$$
y_1(x) = \int_0^x \varphi(s) v(x-s) ds
$$

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